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New Families of Iterative Methods For Multiple Roots For Solving Systems Of Nonlinear Equations

Paul Rai[1], Dr. Sonal Bharti

[1]Research Scholar, Department of Mathematics, Sri Satya Sai University of Technology & Medical Sciences, Sehore, MP(India)

ABSTRACT

We extend to n-dimensional case a known multiple roots family of iterative methods for solving nonlinear equations. This family includes as particular cases some well-known and also some new methods. The proposed families are derived from the modified Newton's method for multiple roots and one-parameter family of Schroder's method.

Keywords: Newton method, Non-linear equations, Iterative method, Order of convergence, Schroder's method

1 INTRODUCTION

This study is attempted to develop a new one-parameter family of Schröder's method for finding multiple roots of nonlinear equations numerically. Further, the researcher has proposed many new cubically convergent families of Schröder-type methods. The proposed families are derived from the modified Newton's method for multiple roots and one-parameter family of Schroder's method. Furthermore, an attempt is made to propose many new families of third-order multipoint iterative methods for multiple roots free from second-order derivative by semi discrete modifications of the above proposed methods. One of the families require two evaluations of the function and one evaluation of its first-order derivative and the other family requires one evaluation of the function and two evaluations of its first-order derivative per iteration. Numerical examples are also presented to demonstrate the performance of the proposed iterative methods.

There is a vast literature on numerical methods for finding multiple roots. All the existing iterative methods for simple roots are linearly convergent for the case of multiple roots. With the advancement of computers, the problem of solving nonlinear equations by numerical methods gained more importance than before.

The purpose of this work is **to construct many new families** of one-point as well as multipoint iterative methods for finding multiple roots of nonlinear equations

$$f(x) = 0, \tag{1.1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a nonlinear sufficiently differentiable function on an open interval I.

If 'r_m' is a multiple root of equation (1.1) with multiplicity 'm' i.e. $f(r_m) = f'(r_m) = \cdots = f^{(m-1)}(r_m) = 0$ and $f^{(m)}(r_m) \neq 0$ (a condition for $x = r_m$ to be root of multiplicity 'm'), then $f(x)$ can be expressed as

$$f(x) = (x - r_m)^m g(x), \tag{1.2}$$

where $g(x)$ is a continuous function with $g(r_m) \neq 0$.

There are two ways of choosing methods for finding multiple roots

(i) when multiplicity 'm' is known,

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(ii) when multiplicity '*m*' is unknown.

Case (i) When '*m*' is known, the well-known Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1.3}$$

used to find 'r_m' for suitably chosen initial approximation 'x_0' generates a sequence of iterates $\langle x_n \rangle$ with erroneous equation given by

$$e_{n+1} = \frac{m-1}{m} e_n + O(e_n^2), \tag{1.4}$$

where $e_n = x_n - r_m$, is an error at n^{th} step. For $m > 1$, the sequence $\langle x_n \rangle$ converges to 'r_m' with the rate of a geometric progression with ratio $\frac{m-1}{m}$. To retain an order of convergence of Newton's method, Rall (essentially due to Schroder) proposed a quadratically convergent modified Newton's method given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}. \tag{1.5}$$

Case (ii) When *m* is unknown, Schröder proposed a quadratically convergent method given by

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}. \tag{1.6}$$

It may be obtained by applying Newton's method to the function $u(x) = \frac{f(x_n)}{f'(x_n)}$, which has simple roots in each multiple zero of $f(x)$.

More recently, the researcher has developed a new family of Newton's method for finding simple roots of a nonlinear equations and is given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{\{f'(x_n)\}^2 + 4\alpha^2\{f(x_n)\}^2}}, \tag{1.7}$$

where $\alpha \in \mathbb{R}$. In (1.7), the sign in denominator should be chosen so that the denominator would be the largest in magnitude.

Here, **the researcher goes a step ahead** by extending the work for the case of multiple roots to construct a modified family of Schroder's method and to propose new cubically convergent families of Schroder-type methods based on power means, when multiplicity '*m*' of a root is known in advance. Thereafter by discretizing second-order derivative, many new second-order derivatives free cubically convergent families of multipoint iterative methods for multiple roots have been acquired. It is also shown that many well-known methods like Super-Halley method, Halley's method, Ostrowski's square root method, Newton-Secant method etc. can be regarded as particular cases of the proposed family.

2.1 DEVELOPMENT OF THE ITERATIVE METHODS

2.1.1 MODIFIED FAMILY OF SCHRÖDER'S METHOD

The family of Newton's method for finding simple root of a nonlinear equation (2.1) can be rewritten as follows:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)\left[1 \pm \sqrt{1 + 4\alpha^2\left\{\frac{f(x_n)}{f'(x_n)}\right\}^2}\right]}. \tag{2.1}$$

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Now consider the factor

$$4\alpha^2\left\{\frac{f(x_n)}{f'(x_n)}\right\}^2. \tag{2.2}$$

Since the scaling parameter 'α' appears in the numerator of (2.2), it is clear that there exits some real values of 'α' such that

$$\left|4\alpha^2\left\{\frac{f(x_n)}{f'(x_n)}\right\}^2\right| << 1, \tag{2.3}$$

holds.

With this assumption, the binomial theorem is applicable in equation (2.1) and one can get the following formula free from square root term as

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{\{f'(x_n)\}^2 + \alpha^2\{f(x_n)\}^2}. \tag{2.4}$$

Further, applying the well-known Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2.5}$$

to the modified function $u_f(x) = \frac{f(x_n)f'(x_n)}{\{f'(x_n)\}^2 + \alpha^2\{f(x_n)\}^2}$ of formula (2.4), one can obtain

$$x_{n+1} = x_n - H(x_n)\frac{f(x_n)f'(x_n)}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}, \tag{2.6}$$

where

$$H(x_n) = \frac{\{f'(x_n)\}^2 + \alpha^2\{f(x_n)\}^2}{\{f'(x_n)\}^2 - \alpha^2\{f(x_n)\}^2}. \tag{2.7}$$

This is a **new one-parameter modified family of Schroder's method** for an equation having multiple root of multiplicity $m > 1$ unknown. It is interesting to note that by ignoring the term in 'α', the method (2.6) reduces to the classical Schroder's method.

A mathematical proof for the order of convergence of the proposed iterative family (2.8) is as under :

Theorem 2.1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function defined on an open interval I, enclosing a multiple zero of f (x) (say $x = r_m \in I$) with multiplicity $m > 1$. Assume that initial guess $x = x_0$ is sufficiently close to 'r_m' then for $\alpha \in \mathbb{R}$, an iteration scheme defined by* (1.13) *has quadratic order of convergence and satisfies the following erroneous equation :*

$$e_{n+1} = -\frac{C_2}{m}e_n^2 + O(e_n^3). \tag{2.8}$$

2.2 SCHRÖDER-TYPE CUBICALLY CONVERGENT ITERATIVE METHODS

Using the modified Newton's formulae (2.6) and (2.8), one can obtain

$$x_{n+1} = x_n - \frac{1}{2}\left[m\frac{f(x_n)}{f'(x_n)} + \left\{H(x_n)\frac{f(x_n)f'(x_n)}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}\right]. \tag{2.9}$$

This is a new one-parameter modified family of Super-Halley type methods for the case of multiple roots of multiplicity $m > 1$ known. It is interesting to note that by ignoring the term 'α', the method (2.9) reduces to Super-Halley method for the case of multiple roots.

The method (2.9) can further be rewritten as

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[\frac{m+\left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}}{2}\right]. \qquad (2.10)$$

Now, the researcher wishes to generalize the formula (2.10) by γ^{th}-power mean. For this, let us take

$$a = m \text{ and } b = \left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}.$$

Approximating the functions in equation (2.10) as follows :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[\frac{m^\gamma+\left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}^\gamma}{2}\right]^{\frac{1}{\gamma}}. \qquad (2.11)$$

It can be seen that the quantities '*a*' and '*b*' are already positive in view of (2.3). This family may be called the γ^{th} – power mean iterative family of Schroder-type methods for multiple roots.

2.3 SPECIAL CASES

It is interesting to note that for different values of 'γ', various well-known methods can be deduced from the formula (2.11) as follows :

(i) For $\gamma = 1$ (arithmetic mean), the formula (2.11) corresponds to cubically convergent Super-Halley type methods for multiple roots

$$x_{n+1} = x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\left[m + \left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}\right]. \qquad (2.12)$$

(ii) Taking $\gamma = -1$ (harmonic mean), the formula (2.11) corresponds to the well-known Halley's type methods for multiple roots

$$x_{n+1} = x_n - \frac{2mH(x_n)f(x_n)f'(x_n)}{m\left[\{f'(x_n)\}^2 - f(x_n)f''(x_n)\right] + H(x_n)\{f'(x_n)\}^2}. \qquad (2.13)$$

(iii) Letting $\gamma = \frac{1}{2}$, the formula (2.11) reduces to a new method for multiple roots given by

$$x_{n+1} = x_n - \frac{1}{4}\frac{f(x_n)}{f'(x_n)}\left[\sqrt{m} + \sqrt{\left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}}\right]^2. \qquad (2.14)$$

(iv) Putting $\gamma = 2$ (root-mean-square), the formula (2.11) reduces to a new method for multiple roots given by

$$x_{n+1} = x_n - \frac{1}{\sqrt{2}}\frac{f(x_n)}{f'(x_n)}\sqrt{m^2 + \left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}^2}. \qquad (2.15)$$

(v) Substituting $\gamma \to 0$ (geometric mean), the formula (2.11) reduces to the well-known Ostrowski's square root type methods given by

$$x_{n+1} = x_n - f(x_n)\sqrt{H(x_n)\frac{m}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}}. \qquad (2.16)$$

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It is interesting to note that by ignoring the term in 'α', the methods (2.12), (2.13) and (2.16) reduce to Super-Halley, Halley's and Ostrowski's square root methods for multiple roots. In addition to this, if $m = 1$ the methods (2.12), (2.13) and (2.16) reduce to Super-Halley, Halley's and Ostrowski's square root methods for simple roots. Next theorem presents a mathematical proof for an order of convergence of the proposed iterative family (2.11).

Theorem 2.2.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function defined on an open interval I, enclosing a multiple zero of f (x) (say $x = r_m \in I$) with multiplicity $m > 1$. Assume that initial guess $x = x_0$ is sufficiently close to 'r_m' then for $\alpha, \gamma \in \mathbb{R}$, an iteration scheme defined by* (2.11) *has cubic order of convergence and satisfies the following erroneous equation :*

$$e_{n+1} = \frac{[m(C_2^2 - 2C_2) - (\gamma C_2^2 + 2\alpha^2)]}{2m^2} e_n^3 + O(e_n^4). \tag{2.17}$$

Some other new cubically convergent iterative methods based on heronian mean, contraharmonic mean, centroidal mean, logarithmic mean etc. can also be obtained from the formula (2.11) as follows :

(vi) New cubically convergent method for multiple roots based on heronian mean is

$$x_{n+1} = x_n - \frac{1}{3}\frac{f(x_n)}{f'(x_n)}\left[m + \sqrt{H(x_n)\frac{m\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}} + H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right]. \tag{2.18}$$

Next theorem presents a mathematical proof for an order of convergence of the proposed iterative scheme (2.18).

Theorem 2.2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function defined on an open interval I, enclosing a multiple zero of f (x) (say $x = r_m \in I$) with multiplicity $m > 1$. Assume that initial guess $x = x_0$ is sufficiently close to 'r_m' then for $\alpha \in \mathbb{R}$, an iteration scheme defined by* (2.18) *has cubic order of convergence and satisfies the following erroneous equation :*

$$e_{n+1} = \frac{C_2^2(3m-2) - 6(mC_3 + \alpha^2)}{6m^2} e_n^3 + O(e_n^4). \tag{2.19}$$

(vii) New cubically convergent method for multiple roots based on contra-harmonic mean is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[\frac{m^2 + \left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}^2}{m + \left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}}\right]. \tag{2.20}$$

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(viii) New cubically convergent method for multiple roots based on centroidal mean is

$$x_{n+1} = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}\left[\frac{m^2 + m\left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - \alpha^2\{f(x_n)\}^2}\right\} + \left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}^2}{m + \left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}}\right]. \tag{2.21}$$

(ix) New cubically convergent method for multiple roots based on logarithmic mean is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[\frac{m - \left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}}{\ln(m) - \ln\left\{H(x_n)\frac{\{f'(x_n)\}^2}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right\}}\right]. \tag{2.22}$$

The cubic order of convergence of above mentioned methods namely, method (2.20), method (2.21) and method (2.22) can be proved on the similar lines as proved in Theorem 2.2.3. Those satisfy the following erroneous equations given by :

$$e_{n+1} = \frac{C_2^2(m-3) - 2(mC_2 + \alpha^2)}{2m^2}e_n^3 + O(e_n^4), \tag{2.23}$$

$$e_{n+1} = \frac{C_2^2(3m-5) - 6(mC_2 + \alpha^2)}{6m^2}e_n^3 + O(e_n^4), \tag{2.24}$$

and

$$e_{n+1} = \frac{C_2^2(3m-1) - 6(mC_2 + \alpha^2)}{6m^2}e_n^3 + O(e_n^4), \tag{2.25}$$

respectively.

2.4 TESTED EXAMPLES

Now, let us present some numerical results obtained by employing iterative methods namely, modified Schroder's method (2.12) (***MSM***), modified Super-Halley method (2.12) (***MSHM***), modified Halley's method (2.13) (***MHM***), modified Ostrowski's square root method (2.16) (***MOM***), method (2.14) (M_1), method (2.15) (M_2), method (2.18) (M_3), method (2.20) (M_4), method (2.21) (M_5) and method (2.22) (M_6) to solve some nonlinear equations with known multiplicity '*m*' given in Table 2.1 and compare those with classical existing methods namely, modified Newton's method (***MNM***), Schroder's method (***SM***), Super-Halley method (***SHM***), Halley's method (***HM***) and Ostrowski's square root method (***OM***). All methods are tested for $|\alpha| = \frac{1}{2}$ and results are summarized in Table 2.2 (Number of iterations), Table 2.3 (Computational order of convergence) and Table 2.4 (Total number of function evaluations) respectively. Computations have been performed by using ***MATLAB***® version 1.5 (***R***2007*b*) in double precision arithmetic. We use $\varepsilon = 10^{-15}$ as a tolerance error. The following stopping criteria are used for computer programmes :

(i) $|x_{n+1} - x_n| < \varepsilon$, *(ii)* $|f(x_{n+1})| < \varepsilon$.

TABLE 2.1 : Test examples, multiplicity (*m*) **and exact root** (r_m)

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Example No.	Examples	Multiplicity (m)	Exact root (r_m)
2.2.1	$x^2 \sin(4x) = 0.$	3	0.0000000000000000
2.2.6	$((x-1)^3 - 1)^{50} = 0.$	50	2.000000000000000
2.2.7	$\exp(x^2 + 7x - 30) - 1)^4 (x - 5) = 0$	4	3.000000000000000
2.2.8	$(1 + \cos(x))\,(\exp(x) - 2)^2 = 0$	2	0.6931471805599454
2.2.9	$\left(\sin(x) - \frac{\sqrt{2}}{2}\right)^2 (x+1) = 0.$	2	0.7853981633974484

TABLE 2.2 : Performance of the methods (number of iterations required), D below stands for divergent

Example No.	Initial guess	MNM	SM	MSM (6.13)	SHM	MSHM (6.30)	HM	MHM (6.31)	OM	MOM (6.34)	M_1 (6.32)	M_2 (6.33)	M_3 (6.41)	M_4 (6.48)	M_5 (6.49)	M_6 (6.50)
2.2.1	−0.28	4	4	4	4	4	4	4	4	4	4	4	4	3	3	3
	0.35	4	5	5	4	4	4	4	4	4	4	4	4	4	3	4
2.2.6	2.5	4	4	4	2	2	2	2	2	2	2	2	2	3	3	2
	3.5	6	6	6	2	2	3	3	3	3	2	3	3	4	3	3
2.2.7	2.9	7	6	6	4	4	4	4	3	3	4	5	4	5	4	4
	3.15	7	10	9	6	6	4	4	4	4	5	D	5	D	D	4
2.2.8	0	6	5	5	3	3	4	3	3	3	3	4	3	4	4	3
	1.5	5	5	5	4	3	4	4	4	3	3	3	3	4	3	3
2.2.9	0.1	5	5	5	4	3	4	4	4	4	4	3	4	4	3	4
	1.25	6	5	5	4	3	4	4	4	4	3	4	3	4	3	3

TABLE 2.3 :Computational order of convergence (COC)

Example No.	**MNM**	**SM**	**MSM** (6.13)	**SHM**	**MSHM** (6.30)	**HM**	**MHM** (6.31)	**OM**	**MOM** (6.34)	**M_1** (6.32)	**M_2** (6.33)	**M_3** (6.41)	**M_4** (6.48)	**M_5** (6.49)	**M_6** (6.50)
2.2.1	2.97	3.00	3.00	3.00	3.00	3.01	3.01	3.00	3.00	3.00	3.00	3.00	2.53	3.00	3.00
	3.00	3.00	3.00	3.00	3.00	3.01	3.00	3.00	3.00	3.00	3.00	3.00	3.00	2.72	2.89
2.2.6	1.99	2.04	2.04	2.58	2.58	2.74	2.74	2.94	2.94	2.84	2.54	2.49	3.02	3.01	3.80
	2.00	2.01	2.05	2.17	2.17	2.83	2.83	2.99	2.99	3.28	3.03	2.92	2.99	3.02	3.17
2.2.7	2.00	2.00	2.00	3.01	3.01	3.00	3.00	2.91	3.00	3.00	3.00	3.00	3.01	3.00	3.00
	2.00	1.98	2.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	...	3.00	...	...	3.00
2.2.8	2.00	1.98	2.00	3.16	4.62	3.00	2.93	2.94	3.07	3.66	2.94	3.95	2.91	3.03	3.43
	2.00	2.00	2.00	3.03	4.40	3.00	3.01	3.01	2.86	3.67	3.33	3.86	2.99	3.47	3.55
2.2.9	1.99	1.98	2.00	3.00	2.72	2.98	2.98	2.99	2.99	3.00	2.51	3.00	2.55	2.66	3.00
	2.00	2.02	2.017	2.53	2.58	3.02	3.01	3.02	3.02	2.81	3.01	2.70	3.02	3.02	2.90

Here, the symbol '...' represents **COC** for a case of divergence.

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TABLE 2.4 : Total number of function evaluations (TNOFE)

Example No.	MNM	SM	MSM (6.13)	SHM	MSHM (6.30)	HM	MHM (6.31)	OM	MOM (6.34)	M_1 (6.32)	M_2 (6.33)	M_3 (6.41)	M_4 (6.48)	M_5 (6.49)	M_6 (6.50)
2.2.1	8	12	12	12	12	12	12	12	12	12	12	12	9	9	9
	8	15	15	12	12	12	12	12	12	12	12	12	12	9	12
2.2.6	8	12	12	6	6	6	6	6	6	6	6	6	9	9	6
	12	18	18	6	6	9	9	9	9	6	9	9	12	9	9
2.2.7	14	18	18	12	12	12	12	9	9	12	15	12	15	12	12
	14	30	27	18	18	12	12	12	12	15	~	15	~	~	12
2.2.8	12	15	15	9	9	12	9	9	9	9	12	9	12	12	9
	10	15	15	12	9	12	12	12	9	9	9	9	12	9	9
2.2.9	10	15	15	12	9	12	12	12	12	12	9	12	12	9	12
	12	15	15	12	9	12	12	12	12	9	12	9	12	9	9

Here, the symbol '~' represents **TNOFE** for a case of divergence.

2.3 GENERALIZED FAMILIES OF MULTIPOINT ITERATIVE METHODS

The main difficulty associated with recently proposed third-order iterative methods is an evaluation of second-order derivative. In recent years, many modifications of Newton's method for multiple roots have been proposed by researchers and the references cited therein. All these modifications are targeted at increasing a local order of convergence with the view of increasing their efficiency index. Some of these multipoint iterative methods are of third-order, while methods proposed by Mir and Rafiq, Neta and Johnson, Neta, Sharma and Sharma, Zhou et al. and Li et al. are of fourth-order. The fourth-order methods proposed by Sharma and Sharma, Zhou et al., Li et al. in and last two formulae in have optimal order of convergence, since they require three evaluations per step, namely one evaluation of function and two evaluations of first-order derivative. All these modifications of Newton's method require the prior knowledge of multiplicity '*m*'. **Here, an attempt has been made to acquire and unify general class of multipoint iterative methods for multiple roots.** The main idea of proposed methods lies in the discretization of second-order derivative involved in γ^{th}-power mean family of Schröder-type iterative methods (1.18). In this section, we have derived two different families by discretizing second-order derivative involved in the family (2.11).

2.3.1 FIRST FAMILY

Expanding the function $f(x_n - \theta u)$ where $\theta \in \mathbb{R} - \{0\}$, but finite, about a point $x = x_n$ with $f(x_n) \neq 0$, we have

$$f(x_n - \theta u) = f(x_n) - \theta u f'(x_n) + \frac{\theta^2 u^2}{2} f''(x_n) + O(u^3). \tag{2.26}$$

Let us take $= \frac{f(x_n)}{f'(x_n)}$, and inserting this into (1.33) and simplifying, one obtains

$$f(x_n)f''(x_n) \approx \frac{2\{f'(x_n)\}^2}{\theta^2 f(x_n)}\{f(x_n - \theta u) - (1-\theta)f(x_n)\}. \tag{2.27}$$

Using this approximate value of $f(x_n)f''(x_n)$ into the formula (1.18), one can have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[\frac{(a_1^*)^\gamma + (b_1^*)^\gamma}{2}\right]^{1/\gamma}, \tag{2.28}$$

where

$$a_1^* = m \text{ and } b_1^* = \left\{H(x_n)\frac{\theta^2 f(x_n)}{(\theta^2 + 2 - 2\theta)f(x_n) - 2f(x_n - \theta u)}\right\}.$$

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An iteration formula (2.28) requires two evaluations function and one first-order derivative per iteration. Further, the family (2.28) forms the basis of proposed family of schemes, which may be described through the following theorem :

Theorem 2.3.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function defined on an open interval I, enclosing a multiple zero of f (x), (say $x = r_m \in I$) with multiplicity $m > 1$. Assume that initial guess $x = x_0$ is sufficiently close to 'r_m' then for $\alpha, \gamma \in \mathbb{R}$*

(i) the iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[\frac{\lambda_1 (a_1^*)^\gamma + \lambda_2 (b_1^*)^\gamma}{2}\right]^{1/\gamma}, \tag{2.29}$$

where $\theta \neq m$, will have third-order convergence in the vicinity of r_m, if

$$\lambda_1 = \mu^{-m}\left[\frac{(\theta - m)}{a_0} + 2\mu^m\right], \tag{2.30}$$

$$\lambda_2 = m^\gamma (m - \theta)\mu^{-m} a_0^{-(1+\gamma)}, \tag{2.31}$$

and

$$\lambda_1 (a_1^*)^\gamma + \lambda_2 (b_1^*)^\gamma > 0. \tag{2.32}$$

(ii) The proposed iterative scheme (2.31) *satisfies the following erroneous equation:*

$$e_{n+1} = \frac{2}{m^2}\left\{2(\theta - m)\left(\frac{\alpha^2 \mu^{-m}}{a_0} + C_3\right) + a_0 \frac{(a_1\theta^3 + a_2\theta^2 + a_3\theta + a_4)}{(m-\theta)\theta^2} C_2^2\right\} e_n^3 + O(e_n^4), \tag{2.33}$$

where

$$\mu = \begin{pmatrix} 1 & \theta \\ & m \end{pmatrix},$$

$$a_0 = \frac{\theta^2}{(2 + \theta^2 - 2\theta - 2\mu^m)},$$

$$a_1 = -(1 + 2m + \gamma),$$

$$a_2 = 2 - 2(1+\gamma)\mu^m + m^2 + 6m + m\gamma + 2\gamma,$$

$$a_3 = -2 + (2 + 4m + 2\gamma)\mu^m - 2m^2 - 8m - 2m\gamma - 2\gamma$$

and

$$a_4 = -(4m - 2m^2 - 2m\gamma)\mu^m + 2m^2 + 4m + 2m\gamma.$$

Special cases:

For different specific values of parameters 'γ' and 'θ', the following various families of multipoint iterative methods for multiple roots can be derived from the family (2.29) :

(i) For $(\gamma, \theta) = (1,1)$, one gets the formula

$$x_{n+1} = x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\left[\lambda_1 m + \lambda_2 H(x_n)\frac{f(x_n)}{f(x_n) - 2f(x_n - u)}\right], \tag{2.34}$$

where

$$\lambda_1 = 2m - m\left(1 - \frac{1}{m}\right)^{-(m-1)}, \tag{2.35}$$

and

$$\lambda_2 = m^2\left[1 - 2\left(1 - \frac{1}{m}\right)^m\right]^2\left(1 - \frac{1}{m}\right)^{-(m-1)}. \tag{2.36}$$

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This is a **new cubically convergent Traub-Ostrowski's type family** of iterative methods for multiple roots. This is a modification over the well-known Traub-Ostrowski's formula for simple roots.

(ii) Taking $(\gamma, \theta) = (-1,1)$, one can get the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \frac{2mH(x_n)f(x_n)}{\lambda_1 H(x_n)f(x_n) + \lambda_2 m[f(x_n) - 2f(x_n - u)]}, \tag{2.37}$$

where

$$\lambda_1 = 2m - m\left(1 - \frac{1}{m}\right)^{-(m-1)}, \tag{2.38}$$

and

$$\lambda_2 = \left(1 - \frac{1}{m}\right)^{-(m-1)}. \tag{2.39}$$

This is a **new cubically convergent Newton-Secant type family** for multiple roots. This is a modification over the Newton-Secant method derived by Kim and Lee.

(iii) Putting $(\gamma, \theta) = (1, -1)$, one can obtain the formula

$$x_{n+1} = x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\left[\lambda_1 m + \lambda_2 H(x_n)\frac{f(x_n)}{5f(x_n) - 2f(x_n + u)}\right], \tag{2.40}$$

where

$$\lambda_1 = 4 + 2m - 5m\left(1 + \frac{1}{m}\right)^{-(m-1)}, \tag{2.41}$$

and

$$\lambda_2 = m^2\left[5 - 2\left(1 + \frac{1}{m}\right)^{m}\right]^2\left(1 + \frac{1}{m}\right)^{-(m-1)}. \tag{2.42}$$

This is a **new cubically convergent formula** for multipoint iterative methods for multiple roots.

Note that the family (2.29) can produce many more new cubically convergent multipoint iterative methods for multiple roots by choosing different values of the parameters.

2.3.2 SECOND FAMILY

Replacing second-order derivative by the following definition :

$$f''(x_n) \approx \frac{f'(x_n) - f'(x_n - \theta u)}{\theta u}, \theta \neq 0 \in \mathbb{R}, \tag{2.43}$$

one gets the following new generalized family as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[\frac{(a_2^*)^\gamma + (b_2^*)^\gamma}{2}\right]^{1/\gamma}, \tag{2.44}$$

where

$$a_2^* = m \text{ and } b_2^* = \left\{H(x_n)\frac{\theta f'(x_n)}{(\theta - 1)f'(x_n) + f'(x_n - \theta u)}\right\}.$$

An iteration formula (2.45) requires one evaluation of function and two of its first-order derivative per iteration. Further, the formula (2.44) forms the basis of proposed family of schemes, which may be described through the following theorem :

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Theorem 2.3.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function defined on an open interval I, enclosing a multiple zero of f (x), (say $x = r_m \in I$) with multiplicity $m > 1$. Assume that initial guess $x = x_0$ is sufficiently close to 'r_m' then for $\alpha, \gamma \in \mathbb{R}$*

(i) the iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[\frac{\delta_1 (a_2^*)^\gamma + \delta_2 (b_2^*)^\gamma}{2}\right]^{1/\gamma}, \tag{2.45}$$

where $\theta \neq m$ and $\theta \neq \frac{2m}{m+1}$, will have third-order convergence in the vicinity of r_m, if

$$\delta_1 = 2m\left[\frac{(m-\theta)^2(-1+\theta+\mu^m) - m\theta(1-\theta)\mu^m}{\theta(m-\theta)[m(-2+\theta)+\theta]}\right]\mu^{-(m-1)}, \tag{2.46}$$

$$\delta_2 = -2\left[\frac{(\theta-1)(m-\theta)+m\mu^m}{\theta[m(-2+\theta)+\theta]}\right] m^{(1+\gamma)} b_0^{-\gamma}\mu^{-(m-1)}, \tag{2.47}$$

and

$$\delta_1 (a_2^*)^\gamma + \delta_2 (b_2^*)^\gamma > 0. \tag{2.48}$$

(ii) The proposed iterative scheme (6.87) *satisfies the following erroneous equation :*

$$e_{n+1} = 2\left\{\frac{2(m-\theta)^2[d_1+d_2\theta+d_3\theta^2] + C_2^2 b_0[b_1+b_2\theta+b_3\theta^2+b_4\theta^3+b_5\theta^4]}{\theta m^2(m-\theta)^2[m(-2+\theta)+\theta]}\right\} e_n^3 + O(e_n^4), \tag{2.49}$$

where

$$\mu = \left(1 - \frac{\theta}{m}\right),$$

$$b_0 = \frac{\theta}{-1+\theta+\mu^{-(m-1)}},$$

$$b_1 = 2m^3(\gamma + m + 2m^2)(1-\mu^m),$$

$$b_2 = -7m^2[1+2m-(1+\gamma)\mu^m] - m^3(1+\gamma)(3-\mu^m) - m^2(5\gamma - 8m\mu^m),$$

$$b_3 = 4m(1+\gamma) + 18m^2 + 13m^3 + m^4 + 7m^2\gamma + m^3\gamma - \mu^m[5m(1+\gamma) + 7m^2 + 5m^2\gamma + 2m^3],$$

$$b_4 = -1-\gamma-7m-5m\gamma-13m^2-2m^2\gamma-3m^3+\mu^m(1+\gamma)(1+m)^2,$$

$$b_5 = 1+3m+2m^2+m(1+\gamma),$$

$$d_1 = 2m\alpha^2(1-\mu^{-1}),$$

$$d_2 = 2C_3 + 2\alpha^2(m+1)\mu^{-m},$$

and

$$d_3 = -2C_3 - C_3 m - 2\alpha^2\mu^{-m}.$$

Special cases:

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For different specific values of parameters 'γ' and 'θ', the following various families of multipoint iterative methods for multiple roots can be derived from the family (2.50).

(I) For $\gamma = 1$ in formula (2.50), one gets the family based on arithmetic mean given by

$$x_{n+1} = x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\left[\delta_1 m + \delta_2 H(x_n)\frac{\theta f'(x_n)}{(\theta-1)f'(x_n)+f'(x_n-\theta u)}\right], \tag{2.50}$$

where

$$\delta_1 = 2\left(1 + \frac{(m-\theta)\mu^{-m}\left((m-\theta)(\theta-1)+m\mu^m\right)}{\theta(\theta(m+1)-2m)}\right), \tag{2.51}$$

and

$$\delta_2 = -\frac{2m\mu^{-m}\left((m-\theta)(\theta-1)+m\mu^m\right)^2}{\theta^2(\theta(m+1)-2m)}. \tag{2.52}$$

Some interesting particular cases of the family (2.50) are as under :

(i) Letting $\theta - 1$ in (2.50), one gets the formula

$$x_{n+1} = x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\left[\delta_1 m + \delta_2 H(x_n)\frac{f'(x_n)}{f'(x-u)}\right], \tag{2.53}$$

where

$$\delta_1 = -2(m-1), \tag{2.54}$$

and

$$\delta_2 = 2m^2\left(1-\frac{1}{m}\right)^{(m-1)}. \tag{2.55}$$

This is a **new cubically convergent family** of iterative methods for multiple roots. This is a modification over the Özban's formula (5) in for simple roots.

(ii) Putting $\theta = \frac{2m}{m+2}$ in equation (2.54), one can get the formula

$$x_{n+1} = x_n - \frac{m}{2}\frac{f(x_n)}{f'(x_n)}\left[\delta_1 + \delta_2 H(x_n)\frac{2f'(x_n)}{(m-2)f'(x_n)+(m+2)f'\left(x_n-\frac{2m}{m+2}u\right)}\right], \tag{2.56}$$

where

$$\delta_1 = -\frac{m^3+4m^2-8+m^2(m-2)\left(\frac{m}{m+2}\right)^{-m}}{2(m+2)}, \tag{2.57}$$

and

$$\delta_2 = -\frac{\left(\frac{m}{m+2}\right)^m\left[(m+2)^2+m(m-2)\left(\frac{m}{m+2}\right)^{-m}\right]^2}{4(m+2)}. \tag{2.58}$$

This is a **new cubically convergent Jarratt-type family** of iterative methods for multiple roots. This is a modification over the well-known Jarratt's method for simple roots.

(II) For $\gamma = -1$ in formula (2.45), one obtains the family based on harmonic mean given by

$$x_{n+1} = x_n - \frac{2m\theta H(x_n)f(x_n)}{\delta_1\theta H(x_n)f'(x_n)+\delta_2 m[(\theta-1)f'(x_n)+f'(x_n-u)]}, \tag{2.59}$$

where

$$\delta_1 = 2m\left[\frac{(m-\theta)^2(-1+\theta+\mu^m)-m\theta(1-\theta)\mu^m}{\theta(m-\theta)\,[m(-2+\theta)+\theta]}\right]\mu^{-(m-1)}, \tag{2.60}$$

and

$$\delta_2 = -\frac{2m\left(1-\frac{\theta}{m}\right)^{-(m-2)}}{m(-2+\theta)+\theta}. \tag{2.61}$$

Some interesting particular cases are as under :

(ii) Substituting $\theta = \frac{1}{2}$ in (2.59), one gets the formula

$$x_{n+1} = x_n - \frac{2mH(x_n)f(x_n)}{\delta_1 H(x_n)f'(x_n)+\delta_2 m[2f'(x_n-0.5w)-f'(x_n)]}, \tag{2.62}$$

where

$$\delta_1 = 2m\left[\frac{(2m-1)^2\left\{-1+2\left(1-\frac{1}{2m}\right)^m\right\}-2m\left(1-\frac{1}{2m}\right)^m}{(2m-1)\,(1-3m)}\right]\left(1-\frac{1}{2m}\right)^{-(m-1)}, \tag{2.63}$$

and

$$\delta_2 = -\frac{4m}{(1-3m)}\left(1-\frac{1}{2m}\right)^{-(m-2)}. \tag{2.64}$$

This is a new cubically convergent family of iterative methods for multiple roots. This is a modification over the well-known iterative method for simple roots.

(ii) If $\theta = 1$ in (2.59) one can get the formula

$$x_{n+1} = x_n - \frac{2mH(x_n)f(x_n)}{\delta_1 H(x_n)f'(x_n)+\delta_2 m f'(x_n-w)}, \tag{2.65}$$

where

$$\delta_1 - 2 - 2m, \tag{2.66}$$

$$\delta_2 = 2\left(1-\frac{1}{m}\right)^{-(m-1)}. \tag{2.67}$$

This is a **new cubically convergent Weerakoon and Fernando-type methods** for multiple roots. It may be noted that by ignoring the term in 'α', the formula (2.67) reduces to well-known cubically convergent Weerakoon and Fernando method for simple roots.

Note that the family (2.45) can produce many more new cubically convergent multipoint iterative methods for multiple roots by choosing different values of the parameters.

2.3.3 TESTED EXAMPLES

Next, an attempt has been made to present some numerical results obtained by employing the iterative methods namely, modified Traub-Ostrowski's method (2.34) (***MTOM***), modified Newton-Secant method (2.37) (***MNSM***), modified Jarratt's method (2.56) (***MJM***), method (2.40) (***MM*$_1$**), method (2.53) (***MM*$_2$**), method (2.62) (***MM*$_3$**), method (2.63) (***MM*$_4$**) to solve some nonlinear equations with known multiplicity '*m*' given in Table 6.1 and compare them with classical existing methods namely, Dong method (***DM***), Victory and Neta method (***VNM***), Traub-Ostrowski's method (***TOM***), Newton-Secant method (***NSM***) and Jarratt's method (***JM***). The results are summarized in Table 2.5 (Number of iterations), Table 2.6 (Computational order of convergence) and Table 2.7 (Total number of function evaluations) respectively. Computations have been performed by using ***MATLAB***® version 2.5 (***R***2007*b*)

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in double precision arithmetic. Here, the formulae are tested for $|\alpha| = \frac{1}{2}$. We use $\varepsilon = 10^{-15}$ as a tolerance error. The following stopping criteria are used for computer programmes :

(i) $|x_{n+1} - x_n| < \varepsilon$, *(ii)* $|f(x_{n+1})| < \varepsilon$.

TABLE 2.5 : Performance of the methods (number of iterations required), D below stands for divergent

Example No.	Initial guess	DM	VNM	TOM	MTOM (6.76)	NSM	MNSM (6.79)	JM	MJM (6.109)	MM_1 (6.82)	MM_2 (6.106)	MM_3 (6.115)	MM_4 (6.118)
1.2.1	−0.28	3	4	4	4	4	4	3	4	4	3	4	3
	0.35	3	4	4	4	4	4	3	4	4	3	4	3
1.2.6	2.5	3	3	3	3	2	3	3	3	3	3	3	3
	3.5	4	4	4	4	4	4	3	4	4	4	4	4
1.2.7	2.9	3	4	5	5	4	4	6	5	7	5	4	3
	3.15	4	4	4	4	4	4	4	4	13	5	4	4
7.2.8	0	4	4	4	3	3	4	3	3	4	4	3	4
	1.5	3	4	3	4	4	3	3	3	4	4	3	4
7.2.9	0.1	4	4	3	4	4	3	3	3	3	4	3	4
	1.25	4	4	3	4	4	3	2	3	4	4	3	4

TABLE 2.6 :Computational order of convergence (COC)

Example No.	DM	VNM	TOM	MTOM (6.76)	NSM	MNSM (6.79)	JM	MJM (6.109)	MM_1 (6.82)	MM_2 (6.106)	MM_3 (6.115)	MM_4 (6.118)
7.2.1	2.62	3.00	3.00	3.00	3.00	3.00	2.33	3.00	3.00	3.10	3.00	2.83
	3.00	3.00	3.00	3.00	3.00	3.00	4.65	3.00	2.99	2.91	3.00	3.00
7.2.6	2.75	2.88	2.96	2.97	2.74	2.89	3.39	2.97	2.95	2.96	2.74	2.81
	2.84	2.82	2.97	2.97	2.83	2.83	3.62	2.97	2.49	2.56	2.83	2.84
7.2.7	3.15	3.00	2.85	2.88	3.00	3.00	4.07	3.01	2.93	3.00	3.00	2.70
	2.84	2.97	2.91	2.92	2.99	2.99	3.91	2.90	2.97	2.79	2.99	3.02
7.2.8	2.95	3.00	3.08	2.25	2.83	2.25	5.16	3.54	2.87	2.95	2.92	2.95
	2.47	2.87	2.75	2.98	2.75	2.98	3.54	3.08	2.87	3.02	2.92	3.02
7.2.9	3.10	2.98	3.00	2.49	2.99	2.49	4.22	2.78	2.77	3.09	2.90	3.12
	2.66	3.01	2.91	3.09	2.94	3.09	6.11	3.11	3.01	3.00	2.79	2.99

TABLE 2.7 : Total number of function evaluations (TNOFE)

Example No.	DM	VNM	TOM	MTOM (6.76)	NSM	MNSM (6.79)	JM	MJM (6.109)	MM_1 (6.82)	MM_2 (6.106)	MM_3 (6.115)	MM_4 (6.118)
7.2.1	9	12	12	12	12	12	9	12	12	9	12	9
	9	12	12	12	12	12	9	12	12	9	12	9
7.2.6	9	9	9	9	6	9	9	9	9	9	9	9
	12	12	12	12	12	12	9	12	12	12	12	12
7.2.7	9	12	15	15	12	12	18	15	14	15	12	9
	12	12	12	12	12	12	12	12	39	15	12	12
7.2.8	12	12	12	9	9	12	9	9	12	12	9	12
	9	12	9	12	12	9	9	9	12	12	9	12
7.2.9	12	12	9	12	12	9	9	9	9	12	9	12
	12	12	9	12	12	9	6	9	12	12	9	12

2.4 DISCUSSION AND CONCLUSIONS

In this chapter, a new family of Schroder-type methods for finding multiple roots of nonlinear equations based on nonlinear means has been proposed. The proposed family of Schröder-type methods (1.18) unifies most of the known third-order iterative methods for solving nonlinear equations and also provide many more unknown processes. Further, many new

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unified and generalized families of third-order multipoint iterative methods free from second-order derivative for multiple roots have been acquired.

It is straight forward to see that per step of these methods require three evaluations of function viz. two evaluations of $f(x)$ and one of $f'(x)$ or one evaluation of $f(x)$ and two of $f'(x)$. In order to obtain an assessment of the efficiency of proposed methods, the efficiency index defined in has been used. For proposed iteration schemes, namely method (2.34), method (1.44), method (2.40), method (2.53), method (2.56), method (2.69) and method (2.72), it is found that $p = 3$ and $d = 3$ yielding $E = \sqrt[3]{3} \approx 1.442$ which is better than $E = \sqrt{2} \approx 1.414$, an efficiency index of Newton's method. The famous Traub-Ostrowski's method and Jarratt's method are examples of optimal fourth-order multipoint iterative methods for simple roots. Unfortunately, the proposed Traub-Ostrowski's type and Jarratt-type methods lose their fourth-order convergence for multiple roots. Numerical examples described here prove that the proposed one-point as well as multipoint iterative methods can compete with any of the existing classical iterative methods for multiple roots. A reasonably close initial guess is required for the multipoint methods to converge. However, this condition applies practically to all the iterative methods for solving equations.

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